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THE GALERKIN METHOD FOR INITIAL VALUE PROBLEMS BASED ON THE PRINCIPLE OF TOTAL VIRTUAL ACTION

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In this paper is presented the derivation of the unconstrained variational statement for initial value problems from the viewpoint of the Principle of Total Virtual Action. Based on the hybrid form of the variational equation, both the Galerkin and the recurrent-Galerkin procedures are developed. They are used to obtain approximate analytical/semi-analytical solutions. Linear and non-linear vibration problems are used to demonstrate the applications. Comparison between the results obtained with initial conditions unconstrained and those with initial conditions constrained is made. Different weighting functions are tried out to confirm the validity of the "hybrid" form of the unconstrained variational statement.

1. INTRODUCTION

The conventional numerical methods widely used for initial value problems are known as *direct integration methods*, in which the solutions of differential equations of motion are obtained in numerical form at *discrete* points by a step-by-step numerical integration procedure. The word "direct" here implies that the integrations are carried out on the differential equations *directly*. Many well-known methods, such as the central difference method, the Houbolt method, the Wilson theta method, the Newmark beta method and the Runge–Kutta method, all fall into this category [1, 2]. Zienkiewicz [3] simply presents these methods in a unified way of finite element approximation in time dimension. On the other hand, the Ritz method and the Galerkin method are known as *direct methods* in the literature, where approximate analytical solutions are obtained *directly* by using variational principles or statements without dealing with the differential equations [4]. The advantage of direct methods lies in the fact that they yield solutions in terms of analytical functions in the finite *continuous* time domain instead of numerical data at *discrete* points of time. To the authors' knowledge, not much attention has yet been given to this issue in the literature.

The application of direct methods to initial value problems became possible only when generalized variational statements for initial value problems could be successfully developed. Based on a *less constrained* variational statement—Hamilton's law of varying action—Bailey [5] obtained approximate solutions for initial value problems via the Ritz method, in which a truncated power series was used as the trial solution. Based on an *unconstrained* variational statement—the extended Hamilton's Principle—Simkins [6] resumed the Ritz method. On the other hand, Leipholz [7] suggested a "hybrid" Galerkin method in obtaining direct solutions for initial value problems, in which the weighting

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functions, although being constrained, could be chosen with greater flexibility. As for non-linear/non-conservative oscillation problems, a Generalized Galerkin Method (GGM) with frequency as an independent unknown was further developed by Chen [8, 9]. This method has been resumed to solve some practical engineering problems by Desai *et al*. [10].

The Principle of Total Virtual Action (PTVA) has been proposed by Chen [11] as a unique foundation for all the constrained/unconstrained variational statements. The "hybrid" form of the variational statements constructed via PTVA can be used to support the Galerkin method in obtaining approximate solutions for initial boundary value problems. In this paper, the procedure of construcing the "hybrid" form of unconstrained variational statement via PTVA for initial value problems will be outlined. Both linear and non-linear oscillators are used to show the application of the Galerkin method in obtaining approximate analytical solutions. In addition, the development of a recurrent-Galerkin calculation procedure will be described. A discussion and conclusions on the proposed procedures are also included.

2. PRINCIPLE OF TOTAL VIRTUAL ACTION AND HYBRID FORM OF UNCONSTRAINED VARIATIONAL STATEMENT

Similar to the Principle of Virtual Work in Statics, being proposed as an axiom in dynamics, the Principle of Total Virtual Action for initial value problems states that:

For a given initial value problem, the actual motion path is such that for any unconstrained virtual displacements the total virtual action of the system vanishes.

In order to provide a clear physical interpretation of the principle, a single-degree-of-freedom system with q being the generalized co-ordinate is considered. Given the initial values q_0 and v_0 at t_0, $\tilde{q}$ is assumed to be the trial solution in a time interval (t_0, t_1), where the prescribed initial conditions are not required to be satisfied. Given an arbitrary unconstrained virtual motion deviation $\delta\tilde{q}$ from $\tilde{q}$ (being unconstrained, $\delta\tilde{q}$ is not required to satisfy $\delta\tilde{q}(t_0) = 0$ and $\delta\tilde{q}(t_1) = 0$), the *total* virtual action of the system in a finite time interval (t_0, t_1) due to the virtual displacement would consist of *three* parts:

(i) Virtual Hamilton's Action in the time inverval (t_0, t_1):

$$\delta A_H = \delta \int_{t_0}^{t_1} L \, \mathrm{d}t = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \dot{\tilde{q}}} \delta\dot{\tilde{q}} + \frac{\partial L}{\partial \tilde{q}} \delta\tilde{q} \right) \mathrm{d}t, \tag{1}$$

where $L = T - V$, L is the Lagrangian, $T = \frac{1}{2}m\dot{\tilde{q}}^2$ kinetic energy, and V is the potential energy of the system.

(ii) Virtual Action due to external forces in the time interval (t_0, t_1):

$$\delta A_f = \int_{t_0}^{t_1} f\delta\tilde{q} \, \mathrm{d}t, \tag{2}$$

where f is a non-potential force since the potential ones are already considered in potential energy V.

(iii) Virtual Action at the time termini. This is the most subtle part of the virtual action. It can be written as

$$\delta A_t = \{m\dot{\tilde{q}}(t_0) - m[\dot{\tilde{q}}(t_0) - v_0]\}\delta\tilde{q}(t_0) + m\delta\dot{\tilde{q}}(t_0)[\tilde{q}(t_0) - q_0] - \{m\dot{\tilde{q}}(t_1) - m[\dot{\tilde{q}}(t_1) - v_1]\}\delta\tilde{q}(t_1) - m\delta\dot{\tilde{q}}(t_1)[\tilde{q}(t_1) - q_1], \tag{3a}$$

where the first terms can be considered as the "gain" of the virtual action due to $\delta\tilde{q}(t_0)$ and $\delta\dot{\tilde{q}}(t_0)$ at t_0, and the other two terms the "loss" due to $\delta\tilde{q}(t_1)$ and $\delta\dot{\tilde{q}}(t_1)$ at t_1. Here $m[\dot{\tilde{q}}(t_0) - v_0]$ and $m[\dot{\tilde{q}}(t_1) - v_1]$ are considered as the "residual linear momentum" at t_0 and t_1, and $[\tilde{q}(t_0) - q_0]$ and $[\tilde{q}(t_1) - q_1]$ the displacement deviations at t_0 and t_1, respectively. Clearly, at t_0 the residual linear momentum and displacement deviations are not supposed to be zero, since $\tilde{q}$ is not required to satisfy the initial conditions. However, since q_1 and v_1 are not prescribed, there is neither a residual linear momentum nor a displacement deviation at t_1. Therefore equation (3a) becomes

$$\delta A_t = mv_0\delta\tilde{q}(t_0) + m\delta\dot{\tilde{q}}(t_0)[\tilde{q}(t_0) - q_0] - m\dot{\tilde{q}}(t_1)\delta\tilde{q}(t_1). \tag{3b}$$

It is noticed that if the trial function $\tilde{q}$ can be chosen such that $\tilde{q}(t_0) = q_0$, and at the termini

$$\delta\tilde{q}(t_0) = 0 \qquad \text{and} \qquad \delta\tilde{q}(t_1) = 0, \tag{4a, b}$$

then δA_t, the virtual action at the termini of the time domain, vanishes. From the viewpoint of the conventional variational theorem, where $\tilde{q}$ and $\delta\tilde{q}$ are related, it is possible to choose a particular trial function $\tilde{q}$ such that condition (4a) is satisfied. However, it is not possible to meet (4b) by choosing the trial function. This is why the conventional Hamilton's principle cannot be used to support the direct method procedure [5].

From the viewpoint of the Principle of Total Virtual Action, the necessary and sufficient condition for $\tilde{q}$ to be the actual motion path is that *for any arbitrary virtual displacement* $\delta\tilde{q}$ *the total virtual action vanishes*:

$$\delta A_T = \delta A_H + \delta A_f + \delta A_t = 0 \tag{5a}$$

or

$$\delta A_T = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial\dot{\tilde{q}}}\,\delta\dot{\tilde{q}} + \frac{\partial L}{\partial\tilde{q}}\,\delta\tilde{q} + f\delta\tilde{q}\right)\mathrm{d}t + mv_0\delta\tilde{q}(t_0) + m\delta\dot{\tilde{q}}(t_0)[\tilde{q}(t_0) - q_0] - m\dot{\tilde{q}}(t_1)\delta\tilde{q}(t_1) = 0, \tag{5b}$$

which is called the *unconstrained variational statement* for the initial value problem.

The significance of "unconstrained" lies in the following two points: (i) the trial solution of the motion $\tilde{q}$ is totally unconstrained, and so it is possible to choose any suitable functions without having to satisfy the initial conditions; (ii) the variation $\delta\tilde{q}$ is also totally unconstrained, and therefore it can simply be denoted by an arbitrary function $\varepsilon\eta(t)$, with ε being a small positive constant, and equation (5b) can then be rewritten in a "*hybrid*" *form of unconstrained variational statement*:

$$\Delta A_T = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial\dot{\tilde{q}}}\,\dot{\eta} + \frac{\partial L}{\partial\tilde{q}}\,\eta + f\eta\right)\mathrm{d}t + mv_0\eta(t_0) + m\dot{\eta}(t_0)[\tilde{q}(t_0) - q_0] - m\dot{\tilde{q}}(t_1)\eta(t_1) = 0, \tag{6}$$

The word "hybrid" means that $\eta(t)$ can be any arbitrary function which is totally unrelated to the trial function $\tilde{q}$.

Equations (5b) or (6) serve as the governing variational equations for initial value problems in such a *direct* way that the differential equation of the motion and the initial conditions are embedded *implicitly*. An alternative expression is obtained by integrating equation (5b) by parts:

$$\delta A_T = \int_{t_0}^{t_1} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\tilde{q}}}\right) + \frac{\partial L}{\partial \tilde{q}} + f \right) \delta\tilde{q} \, \mathrm{d}t - m[\dot{\tilde{q}}(t_0) - v_0]\delta\tilde{q}(t_0) + m\delta\dot{\tilde{q}}(t_0)[\tilde{q}(t_0) - q_0] = 0, \quad (7)$$

and its corresponding "hybrid" form

$$\Delta A_T = \int_{t_0}^{t_1} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\tilde{q}}}\right) + \frac{\partial L}{\partial \tilde{q}} + f \right) \eta \, \mathrm{d}t - m[\dot{\tilde{q}}(t_0) - v_0]\eta(t_0) + m\dot{\eta}(t_0)[\tilde{q}(t_0) - q_0] = 0, \quad (8)$$

where the differential equation of motion and initial conditions are seen to be coupled *explicitly*. Clearly, the "hybrid" form given in equation (6) or equation (8) can be used as a foundation of the Galerkin method simply by treating the arbitrary function η as the weighting function.

3. THE GALERKIN PROCEDURE FOR AN ANALYTICAL SOLUTION

To show the potential application of the Galerkin method based on the PTVA to initial value problems for obtaining approximate *analytical* solutions, two simple examples are used to demonstrate the solution procedure.

3.1. EXAMPLE 1

Consider a one-DOF linear mass–spring system with mass m, and spring stiffness k. Letting $m = 1$, and $k = 4\pi^2$, the Lagrangian becomes

$$L = \tfrac{1}{2}m\dot{\tilde{q}}^2 - \tfrac{1}{2}k\tilde{q}^2 = \tfrac{1}{2}\dot{\tilde{q}}^2 - 2\pi^2\tilde{q}^2. \quad (9)$$

Substituting equation (9) into equation (8) yields the "hybrid" unconstrained variational statement:

$$\Delta A_T = \int_{t_0}^{t_1} (-4\pi^2\tilde{q} - \ddot{\tilde{q}})\eta \, \mathrm{d}t - [\dot{\tilde{q}}(t_0) - v_0]\eta(t_0) + [\tilde{q}(t_0) - q_0]\dot{\eta}(t_0) = 0. \quad (10)$$

Ignoring the inital conditions, $\tilde{q} = A\cos\omega t$ is assumed to be the trial periodic solution and the time interval is chosen to be one period $(\tau, \tau + 2\pi/\omega)$. Equation (10) then becomes

$$\Delta A_T = \int_{\tau}^{\tau + 2\pi/\omega} (-4\pi^2 A\cos\omega t + A\omega^2\cos\omega t)\eta \, \mathrm{d}t$$

$$- [-A\omega\sin\omega\tau - v_0]\eta(\tau) + [A\cos\omega\tau - q_0]\dot{\eta}(\tau) = 0, \quad (11)$$

where in addition to the unknowns A and ω, the starting point $t_0 = \tau$ is also treated as an unknown. If the three weighting functions are chosen to be 1, $\sin\omega t$ and $\cos\omega t$, respectively, the three algebraic equations are as follows:

(i) for $\eta_1 = 1$,

$$v_0 + A\omega\sin\omega\tau = 0; \quad (12a)$$

(ii) for $\eta_2 = \sin \omega t$,

$$v_0 \sin \omega t + A\omega - q_0 \omega \cos \omega\tau = 0; \tag{12b}$$

(iii) for $\eta_3 = \cos \omega t$,

$$(-4\pi^2 A + A\omega^2)(\pi/\omega) + v_0 \cos \omega\tau + q_0 \omega \sin \omega\tau = 0. \tag{12c}$$

Solving the above simultaneous equations yields

$$\omega = 2\pi, \qquad A = \sqrt{\left(\frac{v_0}{2\pi}\right)^2 + q_0^2}, \qquad \tau = \frac{1}{2\pi} \tan^{-1}\left(-\frac{1}{2\pi}\frac{v_0}{q_0}\right). \tag{13a–c}$$

If the initial values q_0 and v_0 are considered to be given at $t = 0$, then the solution can just be expressed as $\tilde{q}(t) = A \cos \omega(t + \tau)$.

In general, for initial value problems having periodic solutions, the trial solution may be assumed in the form of Fourier series:

$$\tilde{q} = A_0 + \sum_{n=1}^{N} (A_n \cos n\omega t + B_n \sin n\omega t), \tag{14}$$

with $2N + 1$ unknown coefficients (A_n and B_n) and unknown frequency ω. The time interval of the integration may be set to $(0, 2\pi/\omega)$. Theoretically, $2N + 2$ algebraic equations could be formulated by choosing $2N + 2$ weighting functions for solving all the unknowns.

3.2. EXAMPLE 2

Consider the free oscillation of a non-linear Duffing's oscillator with $m = 1$ and $k = 1$, the Lagrangian being

$$L = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}q^2 - \tfrac{1}{4}\varepsilon q^2 \qquad (\text{with } \varepsilon \ll 1). \tag{15}$$

The "*hybrid*" *form of unconstrained variational statement* (8) in this case becomes

$$\Delta A_T = \int_{t_0}^{t_1} (-\varepsilon\tilde{q}^3 - \tilde{q} - \ddot{\tilde{q}})\eta \, \mathrm{d}t - [\dot{\tilde{q}}(t_0) - v_0]\eta(t_0) + [\tilde{q}(t_0) - q_0]\dot{\eta}(t_0) = 0. \tag{16}$$

Let the initial conditions be

$$q(0) = q_0, \qquad \dot{q}(0) = v_0 = 0. \tag{17a, b}$$

Assume that the trial solution is

$$\tilde{q} = A \cos \omega t + B \cos 3\omega t, \tag{18}$$

which is *not* required to satisfy the initial displacement condition (17a) at this stage. Choosing the integration interval to be $(0, \pi/\omega)$, and substituting equation (18) into equation (16), yields

$$\int_0^{\pi/\omega} \{A(1 - \omega^2) \cos \omega t + B(1 - 9\omega^2) \cos 3\omega t + \varepsilon[A^3 \cos^3 \omega t + 3A^2 B \cos^2 \omega t \cos 3\omega t$$

$$+ 3AB^2 \cos \omega t \cos^2 3\omega t + B^3 \cos^3 3\omega t]\}\eta \, \mathrm{d}t + [q_0 - (A + B)]\dot{\eta}(0) = 0, \tag{19}$$

in which there are three unknowns, A, B and ω. If the three weighting functions are chosen to be $\cos \omega t$, $\cos 3\omega t$ and $\sin \omega t$, respectively, three algebraic equations can be obtained as follows:

(i) for $\eta_1 = \cos \omega t$,

$$\omega^2 = 1 + \tfrac{3}{4}\varepsilon A^2 + \tfrac{3}{4}\varepsilon AB + \tfrac{3}{2}\varepsilon B^2; \tag{20a}$$

(ii) for $\eta_2 = \cos 3\omega t$,

$$9B\omega^2 = B + \varepsilon(\tfrac{1}{4}A^3 + \tfrac{3}{2}A^2B + \tfrac{3}{4}B^3); \tag{20b}$$

(iii) for $\eta_3 = \sin \omega t$,

$$A + B = q_0. \tag{20c}$$

Using expressions (20a) for ω^2 in equation (20b) shows that B should be in the form of $\varepsilon\beta$. Substituting $B = \varepsilon\beta$ and $A = q_0 - \varepsilon\beta$ into equation (20b), and omitting $O(\varepsilon^2)$ terms, gives

$$\beta = \tfrac{1}{32}q_0^3. \tag{21}$$

Consequently, we have

$$A = q_0 - \tfrac{1}{32}\varepsilon q_0^3, \qquad B = \tfrac{1}{32}\varepsilon q_0^3, \qquad \omega^2 = 1 + \tfrac{3}{4}\varepsilon q_0^2. \tag{22a–c}$$

The approximate solution of the Duffing's oscillator is thus obtained as [8]

$$\tilde{q} = (q_0 - \tfrac{1}{32}\varepsilon q_0^3)\cos \omega t + \tfrac{1}{32}\varepsilon q_0^3 \cos 3\omega t. \tag{23}$$

For non-linear oscillations, the simultaneous algebraic equations are non-linear. The approximate procedure shown in the above example is applicable only for *weakly* non-linear cases ($\varepsilon \ll 1$). For highly non-linear cases, a numerical technique should be used for solving the series of non-linear simultaneous algebraic equations.

4. A RECURRENT-GALERKIN PROCEDURE FOR A SEMI-ANALYTICAL SOLUTION

For general initial value problems, the procedure proposed in the last sections faces two difficulties: (i) the motion may not be periodic, and so the trial solution cannot be assumed in a simple form; (ii) if the time domain concerned is large, it is almost impossible to obtain an analytical expression for the whole domain.

To overcome the first difficulty, the trial solution could be assumed in the form of a truncated Taylor's series [6]

$$\tilde{q}(t) = X_1 + X_2 t + \sum_{i=3}^{N} X_i t^{(i-1)}, \tag{24}$$

with the initial conditions *unconstrained*. The N unknown coefficients X_i in equation (24) could then be determined by solving N Galerkin algebraic equations, constructed by using the governing variational equation (8) with N suitable weighting functions. To deal with the second difficulty, the time domain concerned could be divided into a number of finite intervals $(0, \Delta)$, $(\Delta, 2\Delta)$, $(2\Delta, 3\Delta)$, etc., and the approximate solutions could then be determined for each segment of time. When the solution for the first segment is obtained, the displacement and the velocity at the end of the segment can be used as the initial conditions for the following one. By carrying on with this *recurrent procedure*, a "piecewise" semi-analytical solution can be obtained for the whole time domain. In this

section, the two examples (a free/forced linear mass–spring system and a forced Duffing's oscillator) are used to demonstrate the newly developed *recurrent-Galerkin procedure* for obtaining semi-analytical solutions.

4.1. RECURRENT-GALERKIN PROCEDURE

Consider, a one-DOF linear mass–spring system, with m the mass, k the spring stiffness, c the damping coefficient, and $F(t)$ the force function. The "hybrid" form of unconstrained variational statement (8) can be written as

$$\Delta A_T = \int_{t_0}^{t_1} (-m\ddot{\tilde{q}} - c\dot{\tilde{q}} - k\tilde{p} + F(t))\eta \, \mathrm{d}t - m(\dot{\tilde{q}}(t_0) - v_0)\eta(t_0) + m(\tilde{q}(t_0) - q_0)\dot{\eta}(t_0) = 0; \tag{25a}$$

i.e.,

$$\int_{t_0}^{t_1} (m\ddot{\tilde{q}} + c\dot{\tilde{q}} + k\tilde{q})\eta \, \mathrm{d}t + m\dot{\tilde{q}}(t_0)\eta(t_0) - m\tilde{q}(t_0)\dot{\eta}(t_0) = \int_{t_0}^{t_1} F(t)\eta \, \mathrm{d}t + mv_0\eta(t_0) - mq_0\dot{\eta}(t_0). \tag{25b}$$

For solving N unknown coefficients X_i, N weighting functions are chosen to be

$$\eta_j = t^{(j-1)}, \qquad j = 1, 2, \ldots, N. \tag{26}$$

Substituting equations (24) and (26) into equation (25b), and carrying out the integrations in segment $(0, \Delta)$, leads to N *linear algebraic* equations

$$\sum_{i=1}^{N} A_{ji} X_i = B_j, \qquad j = 1, \ldots, N, \tag{27}$$

where A_{ji} is an $N \times N$ matrix:

$$A_{ji} = m \frac{(i-1)(i-2)}{i+j-3} \Delta^{i+j-3} + c \frac{i-1}{i+j-2} \Delta^{i+j-2} + k \frac{1}{i+j-1} \Delta^{i+j-1}, \tag{28a}$$

with

$$A_{12} = A_{12} + m, \qquad A_{21} = A_{21} - m, \tag{28b}$$

and

$$B_j = \int_0^{\Delta} F(t)\eta_j \, \mathrm{d}t, \tag{29a}$$

with

$$B_1 = B_1 + mv_0, \qquad B_2 = B_2 - mq_0. \tag{29b}$$

A FORTRAN program (program A with initial conditions *unconstrained*) has been developed for solving the series of N linear algebraic equations. As for the integration in equation (29a), a Simpson integration method with an accuracy control parameter EPS is used. Once the approximate solution in the time interval $(0, \Delta)$ is obtained, both $\tilde{q}(\Delta)$ and $\dot{\tilde{q}}(\Delta)$ become the initial conditions for the subsequent interval $(\Delta, 2\Delta)$. The process

TABLE 1

Results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$ *by program A with initial conditions unconstrained*: $\Delta = 0{\cdot}7855$, $N = 6$, $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$

M	t	$\tilde{q}(t)$	$\dot{\tilde{q}}(t)$	$\ddot{\tilde{q}}(t)$	$R(t)$	$q(t)$
1	0·000000	1·000000	−0·000009	−0·999816	0·000180	1·000000
	0·196375	0·981274	−0·187648	−0·906225	−0·000010	0·981251
	0·392750	0·927686	−0·354186	−0·786009	0·000003	0·927642
	0·589125	0·843851	−0·495047	−0·645833	−0·000001	0·843790
	0·785500	0·735153	−0·606966	−0·492362	0·000005	0·735078
2	0·785500	0·735153	−0·606939	−0·492870	−0·000494	0·735078
	0·981875	0·607490	−0·687995	−0·332266	0·000026	0·607404
	1·178250	0·467013	−0·737461	−0·172031	−0·000003	0·466920
	1·374625	0·319883	−0·755935	−0·017513	−0·000003	0·319787
	1·571000	0·172042	−0·745063	0·125945	−0·000038	0·171947
3	1·571000	0·172041	−0·745024	0·125231	−0·000738	0·171947
	1·767375	0·029010	−0·707475	0·254019	0·000039	0·028919
	1·963750	−0·104289	−0·646561	0·362908	−0·000006	−0·104373
	2·160125	−0·223663	−0·566356	0·450202	−0·000004	−0·223737
	2·356500	−0·325751	−0·471269	0·514207	−0·000051	−0·325813
4	2·356500	−0·325752	−0·471240	0·513703	−0·000545	−0·325813
	2·552875	−0·408084	−0·365947	0·554492	0·000029	−0·408133
	2·749250	−0·469111	−0·255045	0·571124	−0·000005	−0·469145
	2·945625	−0·508185	−0·143100	0·565423	−0·000002	−0·508205
	3·142000	−0·525521	−0·034317	0·539213	−0·000035	−0·525526

is repeated for all of the time intervals. By using this recurrence procedure, a *piece-by-piece analytical solution* of the initial value problem can be obtained for any required time domain. Case studies of different system parameters (m, c, k) and initial conditions by using different orders of polynomial (N) and basic time intervals Δ have been carried out and are reported in [12]. As an example, the approximate solution of $\tilde{q}$, and $\dot{\tilde{q}}$ for the system $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$ is given in Table 1 and plotted in Figure 1. The squares are the conjoint

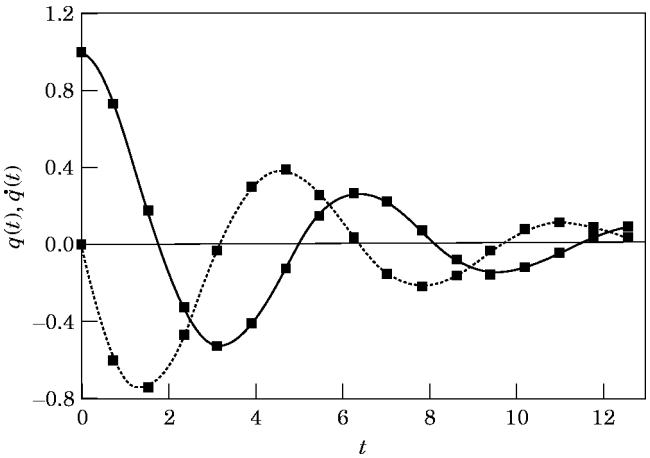

Figure 1. The solution of the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$, with $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$ and $\Delta = 0{\cdot}7855$. ———, $q(t)$; ----, $\dot{q}(t)$.

TABLE 2

Comparison of results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$ *with different values of* N: $\Delta = 0{\cdot}7855$, $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$

		$N = 5$		$N = 6$		$N = 7$	
M	t	$\tilde{q}(t)$	$R(t)$	$\tilde{q}(t)$	$R(t)$	$\tilde{q}(t)$	$R(t)$
1	0·000000	0·999962	−0·012066	1·000000	0·000180	1·000000	0.000061
	0·196375	0·981280	−0·000889	0·981274	−0·000010	0·981274	−0·000004
	0·392750	0·927673	0·000946	0·927686	0·000003	0·927686	0·000002
	0·589125	0·843845	−0·000547	0·843851	−0·000001	0·843851	−0·000001
	0·785500	0·735144	0·001288	0·735153	0·000005	0·735153	−0·000003
2	0·785500	0·735113	−0·010316	0·735153	−0·000490	0·735153	0·000036
	0·981875	0·607485	−0·000679	0·607490	0·000026	0·607490	−0·000002
	1·178250	0·466991	0·000753	0·467013	−0·000003	0·467013	0·000001
	1·374625	0·319867	−0·000442	0·319883	−0·000003	0·319883	−0·000001
	1·571000	0·172024	0·000991	0·172042	−0·000038	0·172043	−0·000002
3	1·571000	0·172013	−0·003851	0·172041	−0·000740	0·172043	0·000000
	1·767375	0·028993	−0·000184	0·029010	0·000039	0·029010	0·000000
	1·963750	−0·104311	0·000234	−0·104290	−0·000006	−0·104289	0·000000
	2·160125	−0·223686	−0·000143	−0·223660	−0·000004	−0·223663	0·000000
	2·356500	−0·325770	0·000275	−0·325750	−0·000051	−0·325750	0·000000
4	2·356500	−0·325760	0·002808	−0·325750	−0·000550	−0·325750	−0·000026
	2·552875	−0·408103	0·000270	−0·408080	0·000029	−0·408083	0·000002
	2·749250	−0·469122	−0·000263	−0·469110	−0·000005	−0·469110	−0·000001
	2·945625	−0·508196	0·000147	−0·508190	−0·000002	−0·508185	0·000001
	3·142000	−0·525528	−0·000386	−0·525520	−0·000035	−0·525520	0·000001

points between the neighboring segments, numbered by M. Since, for any segment, $\tilde{q}(t)$ in the form of equation (24) is available, the velocity and acceleration *at any point in the time domain* can be calculated by using the analytical expressions derived from equation (24). This is obviously one of the advantages of the direct method over the conventional numerical integration methods. Moreover, to estimate the accuracy of the solution, a *residue function* is defined, as

$$R(t) = m\ddot{\tilde{q}} + c\dot{\tilde{q}} + k\tilde{q} - F(t), \tag{30}$$

and this is also listed in Table 1. The smaller the $R(t)$, the better the motion equation is being approximately satisfied. For comparison, the exact solution $q(t)$, is also given in the last column of Table 1.

As expected, increasing N and/or reducing the length of segments of time improves the accuracy substantially (see Tables 2 and 3).

The solutions obtained are *not piecewise*, as shown in Table 2 when N is smaller or in Table 3 when Δ is larger. This is due to the release of the initial conditions in the governing equation (25a), and so the solutions of the coefficients X_1 and X_2 are not usually equal to q_0 and v_0, respectively. Hence, there are *discontinuities* in displacement as well as in velocity at all conjoint points between any two neighboring segments. The discontinuities can be removed by assuming [5] that

$$\tilde{q}(t) = q_0 + v_0 t + \sum_{i=3}^{N} X_i t^{(i-1)} \tag{31}$$

with the initial conditions *constrained*. Then, equation (25a) is reduced to

$$\int_{t_0}^{t_1} [m\ddot{\tilde{q}} + c\dot{\tilde{q}} + k\tilde{q} - F(t)]\eta \, \mathrm{d}t = 0. \tag{32}$$

For solving the $N-2$ unknown coefficients X_i, only $N-2$ weighting functions are needed:

$$\eta_j = t^{(j-1)}, \qquad j = 1, 2, \ldots, N-2. \tag{33}$$

A FORTRAN program (program B with initial conditions *constrained*) has also been complied for this issue. Piecewise solutions can be obtained following the same recurrent-Galerkin procedure. The solution for the aforementioned case is shown in Table 4. For comparison, in Table 5 are provided two sets of solution obtained by both programs A and B for a forced vibration system $\ddot{\tilde{q}} + 0{\cdot}4\dot{\tilde{q}} + q = 0{\cdot}5 \cos 0{\cdot}5t$.

One might expect that program B would give a better solution, since the initial conditions are strictly satisfied. However, from Table 5, the results of program A are always better than those of program B in the sense of having *smaller residues*, except at the starting points of each segment. If the data at all of the starting points are ignored, as they are redundant, and those at all of the end points of the segments are adopted instead, the discontinuities in the results of program A can then be "smoothed". In this way, a *better* piecewise solution for the problem is resulted. Here, "*better*" means that the solution satisfies the equation of motion better. It is seen that for the program A, the "fuzziness" of initial conditions (unconstrained) is rewarded with a better accuracy for the motion equation, whereas the strict satisfaction of initial conditions (constrained) as in program B receives a "penalty" in the form of a loss of accuracy.

TABLE 3

Comparison of results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$, *with different time intervals*: $N = 6$, $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}1$

	$\Delta = 3{\cdot}142$		$\Delta = 1{\cdot}571$		$\Delta = 0{\cdot}7855$	
t	$\tilde{q}(t)$	$R(t)$	$\tilde{q}(t)$	$R(t)$	$\tilde{q}(t)$	$R(t)$
0·0000	0·996628	−0·143013	0·999979	−0·002977	1·000000	0·000180
0·3927	0·928518	−0·033031	0·927682	0·000139	0·927686	0·000003
0·7855	0·734118	0·007414	0·735147	0·000012	0·735153	0·000005
1·1783	0·465156	0·009590	0·467008	−0·000057	0·467013	−0·000003
1·5710	0·170943	−0·000724	0·172036	−0·000396	0·172042	−0·000038
1·9637	−0·104642	−0·006247	−0·104306	0·000565	−0·104290	−0·000006
2·3565	−0·326163	−0·001211	−0·325769	−0·000091	−0·325750	−0·000051
2·7492	−0·469640	0·005628	−0·469126	−0·000046	−0·469110	−0·000005
3·1420	−0·525564	−0·012027	−0·525538	−0·000683	−0·525520	−0·000035
3·1420	−0·523831	0·072747	−0·525529	0·001105	−0·525520	−0·000130
3·5347	−0·499890	0·017055	−0·499685	−0·000048	−0·499670	−0·000002
3·9275	−0·406237	−0·003751	−0·407173	−0·000011	−0·407170	−0·000005
4·3202	−0·269268	−0·004986	−0·270733	0·000028	−0·270730	0·000001
4·7130	−0·114887	0·000331	−0·115975	0·000180	−0·115980	0·000019
5·1057	0·033263	0·003250	0·032632	−0·000303	0·032618	0·000003
5·4985	0·155576	0·000659	0·155035	0·000048	0·155016	0·000027
5·8912	0·238251	−0·002946	0·237793	0·000026	0·237775	0·000003
6·2840	0·275064	0·006316	0·275013	0·000372	0·274995	0·000019

TABLE 4

Results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$ *by program B with initial conditions constrained*: $\Delta = 0{\cdot}7855$, $N = 6$, $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$

M	t	$\tilde{q}(t)$	$\dot{\tilde{q}}(t)$	$\ddot{\tilde{q}}(t)$	$R(t)$	$q(t)$
1	0·000000	1·000000	0·000000	−0·999953	0·000047	1·000000
	0·196375	0·981274	−0·187650	−0·906224	−0·000010	0·981251
	0·392750	0·927685	−0·354185	−0·785997	0·000014	0·927642
	0·589125	0·843851	−0·495046	−0·645844	−0·000012	0·843790
	0·785500	0·735153	−0·606966	−0·492337	0·000030	0·735078
2	0·785500	0·735153	−0·606966	−0·492460	−0·000093	0·735078
	0·981875	0·607490	−0·687990	−0·332264	0·000030	0·607404
	1·178250	0·467013	−0·737463	−0·172066	−0·000038	0·466920
	1·374625	0·319883	−0·755936	−0·017480	0·000028	0·319787
	1·571000	0·172043	−0·745063	0·125875	−0·000108	0·171947
3	1·571000	0·172043	−0·745063	0·125834	−0·000148	0·171947
	1·767375	0·029010	−0·707467	0·254020	0·000044	0·028919
	1·963750	−0·104288	−0·646563	0·362857	−0·000057	−0·104373
	2·160125	−0·223663	−0·566358	0·450250	0·000043	−0·223737
	2·356500	−0·325750	−0·471268	0·514104	−0·000154	−0·325813
4	2·356500	−0·325750	−0·471268	0·514143	−0·000114	−0·325813
	2·552875	−0·408083	−0·365940	0·554492	0·000032	−0·408133
	2·749250	−0·469110	−0·255047	0·571086	−0·000042	−0·469145
	2·945625	−0·508185	−0·143101	0·565458	0·000033	−0·508205
	3·142000	−0·525520	−0·034318	0·539137	−0·000110	−0·525526

4.2. APPLICATIONS TO A DUFFING OSCILLATOR

For a general forced Duffing oscillator, the "hybrid" form of unconstrained variational statement (8) is

$$\Delta A_T = \int_{t_0}^{t_1} (-\varepsilon k\tilde{q}^3 - k\tilde{q} - c\dot{\tilde{q}} - m\ddot{\tilde{q}} + F(t))\eta \, \mathrm{d}t - m(\dot{\tilde{q}}(t_0) - v_0)\eta(t_0) + m(\tilde{q}(t_0) - q_0)\dot{\eta}(t_0) = 0. \tag{34}$$

By denoting

$$F^{(n)}(\tilde{q}, t) = -\varepsilon k\tilde{q}^3 + F(t), \tag{35}$$

equation (34) can be rewritten as

$$\int_{t_0}^{t_1} (m\ddot{\tilde{q}} + c\dot{\tilde{q}} + k\tilde{q})\eta \, \mathrm{d}t + m\dot{\tilde{q}}(t_0)\eta(t_0) - m\tilde{q}(t_0)\dot{\eta}(t_0)$$

$$= \int_{t_0}^{t_1} F^{(n)}(\tilde{q}, t)\eta \, \mathrm{d}t + mv_0\eta(t_0) - mq_0\dot{\eta}(t_0), \tag{36}$$

which is similar to equation (25b). Substituting expressions (24) and (26) into equation (36), and carrying out the integrations in the segment $(0, \Delta)$ leads to N *non-linear* algebraic equations

$$\sum_{i=1}^{N} A_{ji}^{*} X_i = B_j^{*}(X_1, X_2, \ldots, X_N), \qquad j = 1, \ldots, N, \tag{37}$$

where $A_{ji}^{*} = A_{ji}$ as given in equations (28a) and (28b), and

$$B_j^{*} = B_j + B_j^{(n)}. \tag{38}$$

Here B_j is the same as in equations (29a) and (29b), and

$$B_j^{(n)}(X_1, X_2, \ldots, X_N) = \int_0^{\Delta} (-\varepsilon k \tilde{q}^3)\eta_j \, \mathrm{d}t. \tag{39}$$

Based on the above formulation, a FORTRAN program C has been compiled to solve the non-linear problem by an *iteration* method. During the iteration, the value of $\tilde{q}$ obtained in the previous loop is used in equation (39) for the next loop. The process is repeated until required convergence is reached. Taking a forced Duffing oscillator $\ddot{\tilde{q}} + 0{\cdot}4\dot{\tilde{q}} + \tilde{q} + 0{\cdot}5\tilde{q}^3 = 0{\cdot}5\cos(0{\cdot}5t)$ as an example, and letting $N = 6$ and $\Delta = 1{\cdot}0$, the motion for two different initial conditions are obtained. The results are shown in Figures 2 and 3. It can be seen that, after a short period, the motion paths of the non-linear oscillator converge to the *same* periodic loop.

4.3. ALTERNATIVE WEIGHTING FUNCTIONS

The "hybrid" form of unconstrained variational statement (6) or (8) allows the weighting functions to be totally unlinked to the trial solution. Therefore, other weighting

TABLE 5

Comparison of results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0{\cdot}5\cos 0{\cdot}5t$ *with initial conditions unconstrained or constrained*: $\Delta = 1{\cdot}5710$, $N = 6$, $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$, EPS = 1·0E − 5

		Program A		Program B	
M	t	$\tilde{q}(t)$	$R(t)$	$\tilde{q}(t)$	$R(t)$
1	0·0000	0·999987	−0·002076	1·000000	−0·000319
	0·3927	0·963721	0·000108	0·963721	0·000129
	0·7855	0·865755	−0·000002	0·865764	−0·000156
	1·1783	0·724845	−0·000027	0·724846	0·000112
	1·5710	0·560565	−0·000200	0·560568	−0·000514
2	1·5710	0·560549	−0·003294	0·560568	−0·000762
	1·9637	0·390697	0·000216	0·390701	0·000216
	2·3565	0·229331	−0·000050	0·229347	−0·000276
	2·7492	0·085805	−0·000006	0·085805	0·000212
	3·1420	−0·035437	−0·000237	−0·035433	−0·000723
3	3·1420	−0·035428	0·001183	−0·035433	0·000147
	3·5347	−0·134254	−0·000030	−0·134251	−0·000063
	3·9275	−0·213680	−0·000011	−0·213685	0·000077
	4·3202	−0·278510	0·000017	−0·278512	−0·000053
	4·7130	−0·333853	0·000089	−0·333857	0·000251
4	4·7130	−0·333849	0·001424	−0·333857	0·000391
	4·1057	−0·383906	−0·000139	−0·383912	−0·000112
	5·4985	−0·431097	0·000049	−0·431107	0·000146
	5·8912	−0·475640	−0·000008	−0·475642	−0·000113
	6·2840	−0·515602	0·000152	−0·515606	0·000381

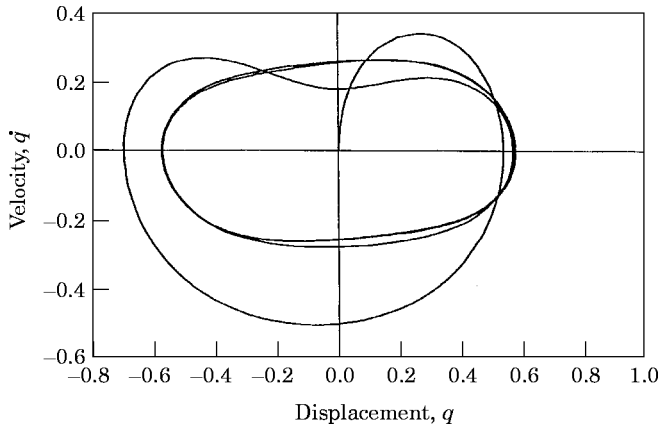

Figure 2. The solution of the case $\ddot{q} + 0{\cdot}4\dot{q} + q + 0{\cdot}5q^3 = 0{\cdot}5\cos 0{\cdot}5t$ with $q_0 = v_0 = 0{\cdot}0$ and $\Delta = 1{\cdot}0$.

functions may be used instead of equation (26). Leipholz [7] used linear combinations of power functions (e.g., $t^3 - t^2$), which were intentionally made to satisfy $\eta(0) = 0$. Here the exponential functions

$$\eta_j = e^{\lambda_j t}, \qquad j = 1, 2, \ldots, N, \tag{40}$$

are used as the alternative weighting functions for the same linear mass–spring system. A FORTRAN program D with initial conditions unconstrained is compiled, based on the new formulations presented in the Appendix. By choosing $\lambda_j = j - 1$ (i.e., choosing the weighting functions to be 1, e^t, e^{2t}, etc.) and using the same input data, the results obtained by program D are as accurate as those obtained by program A. Furthermore, let the various sets of weighting functions be labelled as follows:

$$\text{(a) } t^{j-1}, \qquad \text{(b) } e^{(j-N/2)t}, \qquad \text{(c) } e^{(j-1)t}, \qquad \text{(d) } e^{2(j-1)t}, \qquad \text{(e) } e^{3(j-1)t}. \tag{41}$$

The accuracy of the results obtained by using these weighting functions is presented in Table 6. The maximum absolute residues of the approximate solutions for the case concerned are used to represent the accuracy.

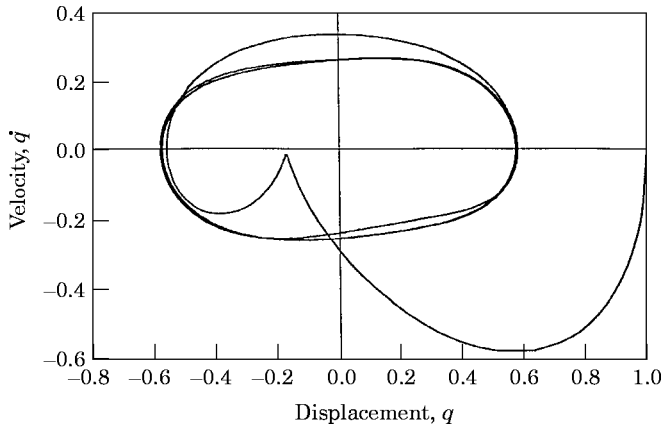

Figure 3. The solution of the case $\ddot{q} + 0{\cdot}4\dot{q} + q + 0{\cdot}5q^3 = 0{\cdot}5\cos 0{\cdot}5t$ with $q_0 = 1{\cdot}0$, $v_0 = 0{\cdot}0$ and $\Delta = 1{\cdot}0$.

TABLE 6

Comparison of results for the case $\ddot{q} + 0{\cdot}4\dot{q} + q = 0$ *with different sets of weighting functions* (*see text*): $\Delta = 1{\cdot}5710,\ N = 6,\ q_0 = 1{\cdot}0,\ v_0 = 0{\cdot}0$

Weighting functions	(a)	(b)	(c)	(d)	(e)
$R(t) <$	7×10^{-4}	6×10^{-4}	4×10^{-4}	8×10^{-4}	2×10^{-3}

There may be an optimization rule for choosing weight functions to yield the best results for the procedure. This is indeed a challenge for future research, to answer how an optimized set of weighting functions can be constructed for a particular problem.

4.4. GENERAL FORMULAE FOR N-DOF SYSTEMS

The "hybrid" form of unconstrained variational statement (8) for one-DOF problems can be further extended to suit N-DOF problems:

$$\Delta A_T = \int_{t_0}^{t_1} E_i \eta_i \, \mathrm{d}t - m_i[\dot{\tilde{q}}_i(t_0) - v_{i0}]\eta_i(t_0) + m_i\dot{\eta}_i(t_0)[\tilde{q}_i(t_0) - q_{i0}] = 0, \tag{42a}$$

where

$$E_i = E_i(q_1, q_2, \ldots, q_N) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\tilde{q}}_i}\right) + \frac{\partial L}{\partial \tilde{q}_i} + f_i, \tag{42b}$$

and q_{i0} and v_{i0} are the given initial conditions for each generalized co-ordinate. In equation (42a) the familiar summation convention over the subscripts ($i = 1, 2, \ldots, N$) is used, and therefore it is a scalar equation. Since the weighting functions η's are arbitrary, for each generalized co-ordinate, we can choose a non-zero weighting function for it, and for the rest we choose zero. Thus, for one weighting function we can have N algebraic equations in the form of equation (42a). Moreover, if M different non-zero weighting functions η_j are to be used, we can have an $M \times N$ *matrix form* of equation (42a) as follows:

$$\Delta A_T^{ij} = \int_{t_0}^{t_1} E_i \eta_j \, \mathrm{d}t - m_i[\dot{\tilde{q}}_i(t_0) - v_{i0}]\eta_j(t_0) + m_i\dot{\eta}_j(t_0)[\tilde{q}_i(t_0) - q_{i0}] = 0. \tag{43}$$

Let the trial solutions, for example, be assumed as

$$\tilde{q}_i(t) = X_{i1} + X_{i2}t + \sum_{j=3}^{M} X_{ij}t^{(j-1)}, \qquad i = 1, 2, \ldots, N \tag{44}$$

with $M \times N$ unknown coefficients X_{ij}, and let M weighting functions be chosen as

$$\eta_j = t^{(j-1)}, \qquad j = 1, 2, \ldots, M. \tag{45}$$

Substituting equations (44) and (45) into equations (43), and carrying out the integrations in segment $(0, \Delta)$ leads to $M \times N$ *algebraic* equations for solving the $M \times N$ unknown coefficients. Based on the recurrent-Galerkin procedure, a FORTRAN program for two-DOF systems is also developed, and various linear and non-linear cases are successfully studied and reported in reference [13].

5. CONCLUSIONS

The Principle of Total Virtual Actions has been proposed as the fundamental axiom for the construction of the unconstrained variational statement for initial value problems. Emphasis is given to the physical interpretation for the third part of the virtual action at the time termini. The obtained "hybrid" form of the governing variational equation (8), as well as equation (43), in which the differential equations of motion and initial conditions are embedded, can be treated as a foundation of the Galerkin method. A typical Galerkin procedure is introduced to obtain approximate periodic solutions in analytical form for a simple linear mass–spring system and a weakly non-linear Duffing oscillator. For more general cases (e.g., transient solutions), a truncated Taylor series is used as a trial solution. By combining the Galerkin method and the recurrent procedure, a semi-analytical solution procedure has been developed. It is seen that the solutions obtained with the initial conditions *unconstrained* are of smaller residues than those with the initial conditions *constrained*. Better accuracy is achieved in terms of satisfying the equation of motion better. The recurrent-Galerkin method has been used in solving general linear mass–spring systems and non-linear oscillation problems. Different weighting functions, i.e., power functions and exponential functions, are also tested to demonstrate the independence between the trial solution and weighting functions in the theory. General formulae for the recurrent-Galerkin procedure for multi-DOF systems are also developed. In conclusion, the Principle of Total Virtual Action has been shown effectively to support the direct methods, and will find wider applications in solving initial boundary value problems.

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APPENDIX: FORMULATION FOR EXPONENTIAL WEIGHTING FUNCTIONS

Define $t = \tau\Delta$, and denote the function

$$\Phi(p, \lambda) = \int_0^1 \tau^p \, e^{\lambda\tau} \, d\tau, \tag{A1}$$

where $p = 0,1, 2, 3, \ldots,$ λ is real. Then, we have following relations:

(i) when $\lambda = 0$,

$$\Phi(p, 0) = \frac{1}{p+1}, \tag{A2}$$

(ii) when $\lambda \neq 0$,

$$\Phi(0, \lambda) = \frac{1}{\lambda}(e^{\lambda} - 1), \qquad p = 0. \tag{A3}$$

When $p > 0$, the recurrence formula is

$$\Phi(p, \lambda) = \frac{1}{\lambda}(e^{\lambda} - p\Phi(p-1, \lambda)), \qquad p > 0. \tag{A4}$$

Therefore, for the alternative weighting functions $e^{\lambda_j t}$, integrating the left side of equation (25b) gives

$$A_{ji} = m(i-1)(i-2)\Phi(i-3, \lambda_j) + c\Delta(i-1)\Phi(i-2, \lambda_j) + k\Delta^2\Phi(i-1, \lambda_j), \tag{A5a}$$

with

$$A_{12} = A_{12} + m, \qquad A_{21} = A_{21} - m; \tag{A5b}$$

and integrating the right side of equation (25b) yields

$$B_j = \int_0^{\Delta} F(t) \, e^{\lambda_j t} \, dt \tag{A6a}$$

with

$$B_1 = B_1 + mv_0, \qquad B_2 = B_2 - mq_0. \tag{A6b}$$